JOURNAL OF COMBINATORIAL THEORY (B) 20, 117–123 (1976)

Reconstruction of a Tree from Its Homomorphic Images and Other Related Transforms

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Communicated by W. T. Tutte

Received October 1, 1974

It is shown that a tree is uniquely reconstructable from its homomorphic images. In addition, the classes of elementary contractions and elementary partitions of a tree are also shown to possess this property. In each case, only the isomorphically distinct transforms of the tree are assumed given for the reconstruction. In the case of elementary contractions the proof uses the reconstruction property of a tree from its endpoint deleted subtrees, which is well-known.

The content of this paper is divided into three parts. Each part shows that a tree T can be reconstructed from the class of graphs obtained by applying a certain transformation ϕ to T. The various $\phi(T)$ considered here are homomorphic images, elementary contractions and elementary partitions, in that order.

The most general reconstruction problem may be stated as given a transformation ϕ (resulting possibly in more than one graph for a given G), when does $\phi(G)$ determine the graph G uniquely, up to an isomorphism. The case where ϕ is the operation of deletion of a vertex has been studied by several authors. The reconstruction conjecture for graphs states that a graph with three or more points is uniquely reconstructable from the collection of its point deleted subgraphs. The conjecture has been verified for a large class of graphs including trees [1], disconnected graphs [4], outer planar graphs [5], and few other classes of graphs. The reconstruction problem for other ϕ's such as those mentioned above were first considered by Sampathkumar and Bhave [3] who have been able to show that the disconnected graphs, with one exception, are reconstructable from their ϕ-images in all three cases.

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In the following, all graphs are assumed simple, i.e., they have no parallel edges or loops.

Let G be a graph. A vertex v is called *pendant* (or, an end vertex) if $\deg(v) = 1$, and a *knot* if $\deg(v) \geqslant 3$. ($\deg(v)$ denotes the degree of v.) For any two nonadjacent vertices x, y the graph obtained by identifying them into a single vertex is called an *homomorphic image*, and is denoted by $G(x, y)$. If T is a tree, then $T(x, y)$ is either a tree (when the distance $d(x, y)$ between x and y equals 2), or a unicyclic graph. In the latter case $T(x, y)$ has one or two less pendant vertices than T according as one or both of x, y are pendant. If G is a unicyclic graph and x is a point on the cycle, then $G(x)$ shall denote the subtree rooted at x. In other words, $G(x)$ is the largest tree subgraph which contains the vertex x, but otherwise disjoint from the cycle. By convention, the root x is not considered an end point of $G(x)$ even if $\deg(x) = 1$.

1. Reconstruction from Homomorphic Images

THEOREM A. *A tree T is uniquely determined from the collection of its isomorphically distinct homomorphic images.*

We begin with few preliminary lemmas.

LEMMA 1. *The number, e, of pendant vertices in T and the degree sequence of T are determined uniquely from $\{T(x, y)\}$.*

Proof. The number of vertices p in T is one plus the number of vertices in a $T(x, y)$. If no $T(x, y)$ contains a cycle, then T is a star and $e = p - 1$ whence the degree sequence of T consists of $(p - 1)$ ones and a $p - 1$. On the other hand, if $T_0 = T(x, y)$ is an homomorphic image that contains the largest cycle, then

$$e = 2 + \#(\text{pendeant vertices in } T_0).$$

The degree sequence of T equals that of T_0 where a two is replaced by a pair of ones. ∎

Let s = degree sequence of T. Any degree sequence obtained by deleting an 1 from s and adding one to another term $\geqslant 2$ shall be denoted by s_1. The degree sequence of T_0 is denoted by s_2. Both s_1 and s_2 have $p - 1$ terms.

LEMMA 2. *If some $G = T(x, y)$ has one of the following forms, then T is uniquely reconstructable*: (i) *G equals a cycle*, (ii) *G contains a 3-cycle in which two vertices have degree 2 and G has degree sequence s_1 or s_2.*

Proof. (i) In this case T is a path.

(ii) First, assume that G has degree sequence s_2. Then remove the edge (a, b) which joins the degree 2 vertices of the cycle and add a new vertex c to G together with the edge (b, c). The resulting graph is the tree T. If the degree sequence of G is like s_1, exactly one of x, y is a pendant vertex and T is obtained by "cutting" the 3-cycle at the point of degree >2 so that the cycle part is replaced by a path of length 3. ▮

In the following we assume that no $T(x, y)$ has a form described in Lemma 2. Thus if v is an end point of T and w is the nearest knot, then $d(v, w) \leqslant 2$. Moreover, if the knowt w is nearest to each of the end points $v_1, v_2, \ldots$ then either all $d(w, v_i) = 1$ or all $d(w, v_i) = 2$.

LEMMA 3. *Suppose $G = T(x, y)$ has degree sequence s_2. If G has a 3-cycle in which exactly one vertex is of degree 2, or contains a 4-cycle where three vertices are of degree 2 each, then T is uniquely reconstructable.*

Proof. In view of the preceding remarks, we have T is obtained by "cutting" the degree 2 vertex of the cycle if G is of the first type, and the degree 2 vertex opposite to the vertex of degree $\geqslant 3$ in the 4-cycle if G is of the second type. ▮

Proof of the theorem. We assume that no homorphic image satisfy Lemmas 2 or 3. We claim that T has two vertices x, y where the distance $d(x, y) = \text{diameter}(T)$ ($=$ the cycle length of T_0) and both are adjacent to vertices of degree $\geqslant 3$. Let $d(u, z) = \text{diameter}(T)$. If u is adjacent to v, $\deg(v) = 2$, then the other vertex w adjacent to v must be a knot. Suppose $d(v', z) = d(u, z) - 1$ and v' adjacent to w. Then v' is a knot in view of Lemmas 2 and 3. Now take x to be an end point adjacent to v'; y can be obtained in a similar way. If x', y' denote the vertices to which x, y are adjacent, then observe that (1) the rooted trees $G(x')$, $G(y')$ where $G = T(x, y)$ are both stars, the root being the vertex of degree >1 if it has three or more vertices, and that (2) the vertex v resulting from the identification of x and y has degree 2 and adjacent to x', y'.

Now choose an homomorphic image G which contains the largest cycle and also contains a vertex v on the cycle having properties (1) and (2). Suppose first that v is not the result of identifying two degree one vertices, i.e., v is a point of the original tree (for example, $v = c$ and $G = T(d, y)$ in Fig. 1). Then there exists another homomorphic image G' which contains a 4-cycle such that two opposite vertices have degree two and G' has degree sequence s_2 (say, $T(a, b)$ in Fig. 1). Clearly, T is obtained by "cutting" the 4-cycle of G' at one of the degree 2 points. Now if no such homomorphic image G' exists, then G is necessarily

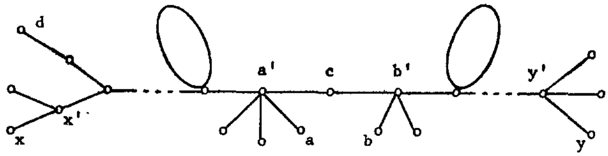

FIG. 1. $T(a, b)$ contains a 4-cycle whose two opposite vertices have degree two each.

of the form $T(x, y)$, x, y as above. The vertex v is the result of identifying x and y, and moreover v is unique. In this case T is obtained by "cutting" the cycle of G at the point v. ∎

2. Reconstruction from Elementary Contractions

An *elementary contraction* (e.c.) graph $G(xy)$ of G is the graph obtained by identifying two adjacent vertices x and y into a single vertex. The object of study in this section is the e.cs. of a tree. If G is a tree graph then each e.c. is a tree. The converse also holds provided G has at least four vertices. There are two 4-point trees which have identical sets of e.cs. These are the 4-path and the 4-star. See Fig. 2 for two more such

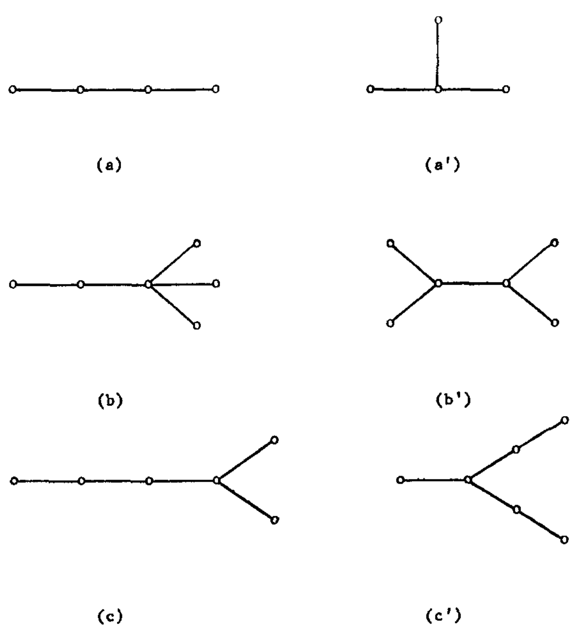

FIG. 2. Three pairs of distinct trees which have identical isomorphism types of elementary contractions. (The trees in (c) and (c′) give same end point deleted subtrees.)

pairs of trees with six points each. We show that except for these any two nonisomorphic trees have a distinct set of elementary contractions. The proof of Theorem B is by giving a method for identifying the subcollection $\{T - v\colon v \text{ pendant}\}$ among all e.cs. and then invoking the theorem of [2], which states that T can be reconstructed from the trees $\{T - v\}$.

Let us define a vertex x to be *outer* if it has degree 2 and lies on a path from some end point v to the knot nearest to v. All degree 2 vertices of a p-path graph are considered to be outer. An elementary contraction $T(xy)$ has one less outer vertices than T if and only if at least one of x, y is outer. In that case, $T(xy) \simeq T - v$ where, say, x is outer and v is the pendant vertex as above; the number of end points in $T(xy)$ is the same as that of T. In general, the number of outer vertices in an e.c. may exceed that of T by an arbitrary amount, but may not be less by more than one.

THEOREM B. *A tree with at least seven vertices is uniquely reconstructable from its isomorphically distinct elementary contractions.*

Proof. We first eliminate some special case. Suppose some $T(xy)$ is a star graph. Then one of the following is true: (i) $T = (k + 2)$-star, (ii) T equals the union of a k_1-star and a k_2-star together with an edge joining the vertices of degree $(k_1 - 1)$ and $(k_2 - 1)$ where both k_1, $k_2 \geqslant 3$ and $k_1 + k_2 = k + 2$, or (iii) T is as in (ii) except for $k_1 = 2$. It is easily seen that each of these trees has a distinct collection of (isomorphically distinct) elementary contractions since $p \geqslant 7$.

Let $t = \min \#(\text{outer vertices in an e.c.})$ and $k = \max \#(\text{end points in an e.c.})$. Suppose there exists an e.c. $T(xy)$ which has k end points and one or more outer vertices. Then T must have at least one outer vertex. If not, the outer vertices in $T(xy)$ must be a result of the contraction, which can happen only if, say, x is an end point and $\deg(y) = 3$. Consequently, T has $k + 1$ end points and this is a contradiction since some e.c. of T, which has diameter 3 or more, will have as many end points. The only tree which has an outer vertex but none of whose k end point e.cs. have an outer vertex is a tree of type (ii) above. However, this case has already been dealt with. (Clearly, such cases do not arise if $t > 0$.) We can summarize all these by saying that T has an outer vertex if and only if some k end point e.c. has an outer vertex. We claim that in this case an e.c. is isomorphic to $T - v$, v pendant, if and only if it has $k - 1$ end points or has k end points and t outer vertices. We prove the sufficiency part. First observe that T itself has k end points and $t + 1$ outer vertices. If $T(xy)$ has $k - 1$ end points then one of x, y is an end point, say x, whence $T(xy) \simeq T - x$. If $T(xy)$ has t outer vertices, then as we have noted earlier $T(xy)$ has the required form. This proves the case when T has outer vertices.

Finally, if T has no outer vertices then we have $T(xy) \simeq T - v$, v pendant, if and only if it has $k - 1$ end points. The theorem is proved. ∎

3. Reconstruction from Elementary Partitions

Finally, we consider the reconstruction of a tree from its elementary partitions. The graph obtained by merging two vertices of G into a single vertex is called an *elementary partition* (e.p.) of G. Each elementary partition is either an elementary contraction or an homomorphic image, according to whether the two vertices were adjacent in G or not. If G has three or more points then it is easily seen that G is connected if and only if each e.p. is a connected graph. Moreover, if the number of vertices $p \geqslant 5$ and G is tree, then each e.p. of G is a tree graph or a unicyclic graph. The converse is also true except for the two cases, G is a 5-cycle or G is a p-star in which two of the $p - 1$ end points have been joined by an edge.

We show that the e.ps. of a tree determine the tree uniquely. There is nothing to prove for $p = 2, 3$. Also the cases $p = 4, 5$ are established by direct verification. (Note that the 3-path has the same e.ps. as does the complete graph on three vertices.) The proof of the general case is actually contained in the proof given for Theorem A.

Theorem C. *Let T be a tree. Then T is uniquely determined, up to isomorphism, from the distinct isomorphism types of its elementary partitions.*

Proof. Assume $p \geqslant 6$. Divide the distinct elementary partitions of T into two classes, C_1 containing the unicyclic graphs and C_2 the tree e.ps. C_1 is empty if and only if T is a star and in this case T is determined. If $C_1 \neq \varnothing$ and some graph in C_1 satisfies the conditions in Lemma 2 or Lemma 3 then we are done. Otherwise, find G in the class C_1 which has degree sequence s_2 and contains a 4-cycle two of whose diagonally opposite vertices are of degree two each. T is obtained by "cutting" the cycle of G at one of these points. However, if no such G exists then find $G = T(x, y)$ as in the proof of Theorem A, and T is reconstructed. ∎

Acknowledgment

We like to express our sincere thanks to the referee for pointing out an error in the proof of Theorem A.

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